

SEMI-ALGEBRAIC GROUPS AND THE LOCAL CLOSURE OF AN ORBIT IN A HOMOGENEOUS SPACE

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ABSTRACT. Let L be a topological group acting on a locally compact Hausdorff space M as a transformation group. Let m be in M . A subset Q of M is called the *local closure* of the orbit Lm if Q is the smallest locally compact invariant subset of M with $m \in Q$. A partition

$$M = \bigcup_{\lambda \in \Lambda} Q_\lambda, \quad Q_\lambda \cap Q_\mu = \emptyset \quad (\lambda \neq \mu)$$

is called an *LC-partition* of M with respect to the L action if each Q_λ is the local closure of Lm for any m in Q_λ .

THEOREM. Let G be a connected Lie group, and let A and B be subgroups of G with only finitely many connected components. Suppose that B is closed. Then the factor space G/B has an LC-partition with respect to the A action.

1. Introduction. Let G be a locally compact topological group, and let A and B be subgroups of G . A subset P of G is said to be (A, B) -invariant if $APB = \{apb; a \in A, p \in P, b \in B\} = P$. In this case the direct product topological group $A \times B$ acts on the underlying space of G as a transformation group by

$$(a, b)g = agb^{-1} \quad \text{for } g \in G, \quad (a, b) \in A \times B,$$

and P is (A, B) -invariant if and only if P is invariant under $A \times B$. For g in G , the double coset AgB is the orbit of the transformation group, passing through g . In this setting, we shall give some definitions.

DEFINITION 1. Let g be in G , and suppose that there exists a minimal, locally compact, (A, B) -invariant subset P containing g . Then the set P is said to be the *local closure* of the double coset AgB .

Since the intersection of two locally compact sets is locally compact, the local closure of AgB is unique, if it exists.

DEFINITION 2. Suppose that G has a partition

$$G = \bigcup_{\lambda \in \Lambda} P_\lambda, \quad P_\lambda \cap P_\mu = \emptyset \quad (\lambda \neq \mu),$$

where Λ is a set of indices, such that for each $\lambda \in \Lambda$, the set $P_\lambda \neq \emptyset$ is the

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local closure of AgB for every g in P_λ . Then the partition (clearly unique) is called the *LC-partition* of G with respect to the pair (A, B) .

Throughout the paper, the identity element of a group in question is always denoted by e ; for a topological group L , the identity component, i.e. the connected component containing e , will be denoted by L_e .

A topological group L is said to be *compactly*, *finitely*, or *countably connected* if the factor group L/L_e is compact, finite, or countable, respectively.

The following theorem will be established in this paper:

THEOREM I. *Let G be a connected Lie group, and let A and B be finitely connected subgroups of G . Then G has an LC-partition with respect to (A, B) .*

Next, we shall consider transformation groups.

DEFINITION 3. Let L be a topological group acting on a locally compact Hausdorff space M as a transformation group. Let m be in M . A subset Q of M is called the *local closure* of the orbit Lm if Q is the smallest locally compact, invariant subset of M with $m \in Q$. Also, a partition

$$M = \bigcup_{\lambda \in \Lambda} Q_\lambda, \quad Q_\lambda \cap Q_\mu = \emptyset \quad (\lambda \neq \mu),$$

is called the *LC-partition*, of M with respect to the L action, if each Q_λ is the local closure of Lm for any m in Q_λ .

Let G be a locally compact group, and B a closed subgroup of G . Let $M = G/B = \{gB; g \in G\}$ denote the factor space, and $\pi: G \ni g \mapsto gB \in M$ the natural map. Let A be a subgroup of G . Then A acts on the homogeneous space M by $a\pi(g) = \pi(ag)$ for $a \in A$ and $g \in G$. Now, it is easy to see that for any $g \in G$, a subset Q of M is the local closure of the orbit $A\pi(g) = \pi(Ag)$ if and only if $\pi^{-1}Q$ is the local closure of the double coset AgB . Therefore from Theorem I we have

THEOREM I'. *Let G be a connected Lie group, and let A and B be finitely connected subgroups. Suppose that B is closed, and let $M = G/B$ be the factor space. Then M has an LC-partition with respect to the A action.*

Throughout the paper, for a subset X of a topological space, we let $\bar{X}$ denote the closure of X .

Let L be a topological group, and let A and B be subgroups of L . If a locally compact subset P of L is (A, B) -invariant, then P is $(\bar{A}, \bar{B})$ -invariant, see (2.1) below. Also if B is finitely connected, then so is $\bar{B}$; see (2.3). Therefore Theorem I' implies Theorem I, i.e. *Theorem I and Theorem I' are equivalent to each other.*

Next, in Theorem I' we let A be an abelian group and B a connected

group. In this case, each local closure has an extremely simple topological structure.

THEOREM II. *Let G be a connected Lie group. Let A be a finitely connected, abelian subgroup and B a closed connected subgroup of G . Then the local closure of any orbit of the transformation group A acting on $M = G/B$ is homeomorphic with the underlying space of a certain abelian Lie group.*

If, in particular, A is a one-parameter subgroup, then for any $m \in M$, the local closure of the orbit Am is homeomorphic with either the real line $\mathbf{R}$ or a toral group.

In order to explain the outline of our proof, we shall introduce a notation of an LC-family of subgroups. When a topological space X is a union of countably many compact subsets, X is said to be σ -compact.

DEFINITION 4. Let G be a locally compact, σ -compact topological group. A set $\mathcal{F}$ of subgroups of G is said to form an LC-family of G if the following conditions are satisfied:

- (1) Any member of $\mathcal{F}$ is closed.
- (2) $G \in \mathcal{F}$, and for any nonempty subset $\mathcal{F}'$ of $\mathcal{F}$, the intersection of all members of $\mathcal{F}'$ is contained in $\mathcal{F}$.

By (2), to any subgroup H of G , we can associate the $\mathcal{F}$ -hull $\mathcal{F}(H)$, which is the smallest member of $\mathcal{F}$ containing H . By (1), $\mathcal{F}(\overline{H}) = \mathcal{F}(H)$.

- (3) If $F \in \mathcal{F}$ and $g \in G$, then $gFg^{-1} \in \mathcal{F}$.
- (4) For F_1 and F_2 in $\mathcal{F}$, the double coset F_1F_2 is a locally compact set.
- (5) Suppose F_1 and F_2 are closed subgroups of G such that $F_1 \supset F_2$ and the factor space F_1/F_2 is compact and totally disconnected. If one of F_1 and F_2 is in $\mathcal{F}$, then so is the other one.

(6) If H is a closed connected subgroup of G , then $\mathcal{F}(H)$ is connected, and the commutator subgroup of H coincides with the commutator subgroup of $\mathcal{F}(H)$.

(7) If H is a compactly connected, abelian subgroup of G , then $\mathcal{F}(H)$ is abelian.

Suppose that G has an LC-family $\mathcal{F}$. Let A and B be subgroups of G , and put $A^* = \mathcal{F}(A)$ and $B^* = \mathcal{F}(B)$. By (3) and (4), every double coset A^*gB^* is locally compact, and $G = \bigcup A^*gB^*$ gives a partition of G into locally compact, (A, B) -invariant subsets. Hence we only have to study the action of $A \times B$ in each A^*gB^* . On the other hand, since $A^* \times B^*$ is a locally compact, σ -compact group, the map

$$\xi_g: A^* \times B^* \ni (\alpha, \beta) \mapsto \alpha g \beta^{-1} \in A^*gB^*$$

is (continuous and) open, see (2.4), and induces a homeomorphism between the factor space $A^* \times B^*/D(g)$ and A^*gB^* , where $D(g)$ is the isotropy

subgroup of $A^* \times B^*$ at g . We see easily that a subset P of A^*gB^* is locally compact if and only if $\xi_g^{-1}P$ is so, and P is (A, B) -invariant if and only if $\xi_g^{-1}P$ is $(A \times B, D(g))$ -invariant. Thus the existence of an LC-partition of G with respect to (A, B) , is reduced to the existence of an LC-partition of $A^* \times B^*$ with respect to $(A \times B, D(g))$ for every $g \in G$.

Next, let us consider the double coset decomposition

$$A^* \times B^* = \cup (A \times B)x D(g).$$

When A and B are closed connected subgroups, $A \times B$ is a normal subgroup of $A^* \times B^*$, and $(A \times B)D(g)$ is a group. Therefore each double coset is a coset. It is easy to see that

$$A^* \times B^* = \cup x \overline{(A \times B)D(g)}$$

gives the LC-partition. When A and B are compactly connected or when A is abelian, etc., we can modify the connected case and can study the $(A \times B) \times D(g)$ action on $A^* \times B^*$ in details using (5), (6) and (7). The results following from the existence of an LC-family are given in Theorem III, §2.

Now the problem boils down to the following:

Which groups have an LC-family?

First we consider $GL(n, \mathbf{R})$. Let $\mathcal{Q}^*$ denote the set of all algebraic groups in $GL(n, \mathbf{R})$. $\mathcal{Q}^*$ is almost an LC-family, but does not satisfy (5), (6). We shall call a subgroup A of $GL(n, \mathbf{R})$ *pre-algebraic* if A is an open subgroup of a suitable algebraic group. Let $\mathcal{Q}$ denote the set of all pre-algebraic groups in $GL(n, \mathbf{R})$. Then $\mathcal{Q}$ is known to be an LC-family. For a subgroup H of $GL(n, \mathbf{R})$, let $\mathcal{Q}(H)$ denote the pre-algebraic hull (= $\mathcal{Q}$ -hull) of H .

A notion of (connected) semi-algebraic groups was introduced by the author in [1]. Here we shall extend it to nonconnected groups. A closed subgroup S of $GL(n, \mathbf{R})$ is said to be *semi-algebraic* if the factor space $\mathcal{Q}(S)/S$ is homeomorphic with a euclidean space $\mathbf{R}^k$. In §3, we shall see that the set $\mathfrak{S}$ of all semi-algebraic groups forms an LC-family.

Let G be a connected Lie group, and $\mathfrak{g}$ its Lie algebra. Then for $g \in G$, the inner automorphism $G \ni x \mapsto gxg^{-1} \in G$ gives rise to an automorphism $\text{Ad}(g)$ of $\mathfrak{g}$, and we have a representation (the adjoint representation)

$$G \ni g \mapsto \text{Ad}(g) \in GL(\mathfrak{g}).$$

If the adjoint group $\text{Ad}(G)$ is semi-algebraic, G is said to be *adjoint semi-algebraic*. In [1], the author proved that for any connected Lie group G , there exists an adjoint semi-algebraic group S containing G as a closed normal subgroup. Then by the nature of our problems, it suffices to consider only adjoint semi-algebraic groups instead of considering general connected Lie groups.

Let G be an adjoint semi-algebraic group. We shall denote $\rho(g) = \text{Ad}(g)$

for $g \in G$. A subgroup H of G is called an *sa-group* if

- (1) $\rho(H)$ is semi-algebraic, and
- (2) H is an open subgroup of $\rho^{-1}\rho(H)$.

In §4, we shall see that the set of sa-groups in G forms an LC-family.

Special cases of our results have been studied in Pukanszky [6] and Goto [1]. In particular, the last part of Theorem II was given in [1], in a slightly weaker form.

2. Locally compact groups and LC-families of subgroups. First let us recall some results on locally compact Hausdorff spaces and locally compact groups (= locally compact topological groups).

Let M be a Hausdorff space, and let Q be a subset of M . If Q is locally compact, then Q is the intersection of a closed set and an open set. If, in particular, M is locally compact, then the converse is also true. Notice that the intersection, but not the union, of finitely many locally compact sets is again locally compact.

Let G be a locally compact group. Then a subgroup H of G is closed if and only if H is locally compact. Suppose that H is a closed subgroup of G . Let $M = G/H$ denote the factor space, and $\pi: G \ni g \mapsto gH \in M$ the natural map. Then a subset Q of M is open, closed, or locally compact if and only if $\pi^{-1}Q$ is so.

(2.1) Let G be a topological group, and let A and B be subgroups of G . Let P be a locally compact subset of G . If P is (A, B) -invariant, then P is $(\bar{A}, \bar{B})$ -invariant.

PROOF. $AP = P$ implies $\bar{A}\bar{P} = \bar{P}$, and we have that $A(\bar{P} - P) = \bar{P} - P$. But $\bar{P} - P$ is a closed set, since P is locally compact. Hence $\bar{A}(\bar{P} - P) = \bar{P} - P$, and with $\bar{A}\bar{P} = \bar{P}$, we have that $\bar{A}P = P$. In a similar way, $P\bar{B} = P$. $\square$

(2.2) Let G be a locally compact group, and let D and E be subgroups of G . Suppose that E is a compactly connected, closed subgroup and the identity component E_e of E is a normal subgroup of G . Then the double coset decomposition

$$G = \cup Eg\overline{DE_e}$$

is the LC-partition of G with respect to (E, D) . Furthermore, each $Eg\overline{DE_e}$ is a closed subset of G .

PROOF. Let $\sigma: G \ni g \mapsto gE_e \in G/E_e$ be the natural homomorphism. We put $Eg\overline{DE_e} = P = P(g)$. Then $\sigma P = \sigma E \cdot \sigma g \cdot \overline{\sigma D}$ is closed, because σE is a compact group. Hence $P = \sigma^{-1}\sigma P$ is closed.

On the other hand, if Q is a locally compact, (E, D) -invariant subset containing g , then Q is (E, DE_e) -invariant, and $Q \supset P(g)$ by (2.1). Hence $P(g)$ is the local closure of EgD . $\square$

(2.3) Let G be a locally compact group, and H a subgroup of G . If H is compactly or finitely connected, then so is $\overline{H}$.

PROOF. Suppose that H is compactly connected. For each h in H , we pick a compact neighborhood $U(h)$ of h in the closure $\overline{H}$ of H . Then $V(h) = U(h) \cap H$ is a neighborhood of h in H . Let φ denote the natural homomorphism

$$H \ni h \mapsto hH_e \in H/H_e.$$

Then $\varphi(V(h))$ is a neighborhood of hH_e in H/H_e . Since H/H_e is compact, there exists a finite set $\{h_1, \dots, h_k\} \subset H$ such that $\bigcup_{i=1}^k \varphi(V(h_i)) = H/H_e$, i.e. $\bigcup_{i=1}^k V(h_i)H_e = H$. We put $C = \bigcup_{i=1}^k U(h_i)$. Then C is a compact set, and $C \cdot \overline{H_e}$ is closed. Since $\overline{H} \supset C\overline{H_e} \supset H$, we have $\overline{H} = C\overline{H_e}$. But since $\overline{H_e}$ is connected, we have $(\overline{H})_e \supset \overline{H_e}$ and $\overline{H} = C(\overline{H})_e$. Hence $\overline{H}$ is compactly connected.

Next, suppose that H is finitely connected, and $H = \bigcup_{i=1}^j a_i H_e$ is the coset decomposition. Then $\overline{H} \supset \bigcup_{i=1}^j a_i \overline{H_e} \supset H$, and $\bigcup_{i=1}^j a_i \overline{H_e}$ is closed. Hence $\overline{H} = \bigcup_{i=1}^j a_i \overline{H_e}$. $\square$

Next, let G be a locally compact, σ -compact group, and suppose that there exists an LC-family $\mathcal{F}$ of subgroups of G , satisfying (1), $\dots$, (7) in §1. Let A and B be subgroups of G . First we shall explain our method to study the $A \times B$ action on G .

We put $\mathcal{F}(A) = A^*$ and $\mathcal{F}(B) = B^*$. By (3) and (4), for any g in G , we have that $A^*(gB^*g^{-1})$ is locally compact, and so is A^*gB^* . Hence the double coset decomposition

$$G = \bigcup_{g \in G} A^*gB^*$$

is a partition of G into locally compact, (A, B) -invariant subsets. In order to study the $A \times B$ action on each A^*gB^* , we need the following known theorem, see e.g. Helgason [4].

(2.4) Let L be a locally compact, σ -compact group, and let M be a locally compact Hausdorff space. Suppose that L acts on M transitively. For m in M , let L_m denote the isotropy subgroup at m : $L_m = \{x \in L; xm = m\}$. Then the map $\xi: L \ni x \mapsto xm \in M$ is (continuous and) open and gives rise to a homeomorphism between the factor space L/L_m and M .

Since A^* and B^* are closed subgroups of G , they are σ -compact, and so is $A^* \times B^*$. On the other hand, $A^* \times B^*$ acts transitively on A^*gB^* and the isotropy group $D(g)$ at g is given by

$$D(g) = \{(\gamma, g^{-1}\gamma g); \gamma \in A^* \cap gB^*g^{-1}\}.$$

By (2.4), the map

$$\xi = \xi_g: A^* \times B^* \ni (\alpha, \beta) \mapsto \alpha g \beta^{-1} \in A^*gB^*$$

induces a homeomorphism between the factor space $(A^* \times B^*)/D(g)$ and A^*gB^* .

Let P be a subset of A^*gB^* . If P is locally compact, then so is $\xi^{-1}P$, and conversely. Suppose that P is (A, B) -invariant. Then for $(\alpha, \beta) \in \xi^{-1}P$ and $(a, b) \in A \times B$, we have

$$\xi((a, b)(\alpha, \beta)) = a\alpha g\beta^{-1}b \in APB = P.$$

Hence $\xi^{-1}P$ is $(A \times B, D(g))$ -invariant. Also it is easy to see that if $\xi^{-1}P$ is $(A \times B, D(g))$ -invariant, then P is (A, B) -invariant. Therefore, ξ^{-1} gives a one-one correspondence between the totality of locally compact, (A, B) -invariant subsets of A^*gB^* , and the set of all locally compact, $(A \times B, D(g))$ -invariant subsets of $A^* \times B^*$.

Suppose that B is closed and consider the map

$$\eta = \eta_g: A^* \times B^* \ni (\alpha, \beta) \mapsto \alpha g \beta^{-1} B \in A^*gB^*/B \subset G/B = M.$$

The map η is also continuous and open, and gives rise to a homeomorphism between the totality of double cosets $(\{e\} \times B) \setminus A^* \times B^*/D(g)$ and A^*gB^*/B .

Before stating the theorem, we shall prove one lemma. For a subgroup H of G , we shall denote by $\mathcal{F}_e(H)$ the identity component of $\mathcal{F}(H)$.

(2.5) Let H be a compactly connected, closed subgroup of G . Then

$$\mathcal{F}(H) = H \cdot \mathcal{F}(H_e), \quad \mathcal{F}(H_e) = \mathcal{F}_e(H).$$

$\mathcal{F}(H)$ is compactly connected, and H_e is a normal subgroup of $\mathcal{F}(H)$.

PROOF. Let N denote the normalizer of H_e in G : $N = \{g \in G; gH_e g^{-1} = H_e\}$. Then $\mathcal{F}(H_e) \subset N$ by (6). For $g \in N$, by (3) we have

$$\mathcal{F}(H_e) = \mathcal{F}(gH_e g^{-1}) = g\mathcal{F}(H_e)g^{-1},$$

and accordingly $\mathcal{F}(H_e)$ is a normal subgroup of N . Since $H \subset N$, we have that $H \cdot \mathcal{F}(H_e)$ is a subgroup of N .

Since H/H_e is compact, there exists a compact subset C of H such that $H = CH_e$. Hence $H \cdot \mathcal{F}(H_e) = C \cdot \mathcal{F}(H_e)$ is a closed subgroup, and we have

$$H \cdot \mathcal{F}(H_e)/\mathcal{F}(H_e) \cong H/H \cap \mathcal{F}(H_e).$$

Since $H \cap \mathcal{F}(H_e) \supset H_e$, the factor group $H/H \cap \mathcal{F}(H_e)$ is compact and totally disconnected. Since $\mathcal{F}(H_e)$ is connected by (6), we see that $H \cdot \mathcal{F}(H_e)$ is compactly connected, and the identity component of $H \cdot \mathcal{F}(H_e)$ is $\mathcal{F}(H_e)$. By (5), $H \cdot \mathcal{F}(H_e) \in \mathcal{F}$, and we have $\mathcal{F}(H) = H \cdot \mathcal{F}(H_e)$. $\square$

THEOREM III. Let G be a locally compact, σ -compact group, with an LC-family $\mathcal{F}$. Let A and B be subgroups of G .

(a) If A and B are compactly connected, then G has an LC-partition with respect to (A, B) .

(b) Suppose that A is a compactly connected, abelian group and B is a closed, connected group. Then for each $g \in G$, we can find a locally compact abelian group $L(g)$, a closed subgroup $L'(g)$ of $L(g)$, and a homeomorphism φ from $\mathcal{F}(A)g\mathcal{F}(B)/B$ onto $L(g)$, such that the image of the local closure of each orbit Am , $m \in \mathcal{F}(A)g\mathcal{F}(B)/B$, is a coset of $L'(g)$.

(c) In (b), if in particular A is a one-parameter subgroup, then the local closure of an orbit Am is homeomorphic either with $\mathbf{R}$ or a certain compact connected abelian group.

PROOF OF (a). By (2.3), $\bar{A}$ and $\bar{B}$ are compactly connected. Hence after this we can suppose that A and B are closed, by (2.1). Then by (2.5), A_e and B_e are normal subgroups of $A^* = \mathcal{F}(A)$ and $B^* = \mathcal{F}(B)$, respectively. Hence $(A \times B)_e = A_e \times B_e$ is a normal subgroup of $A^* \times B^*$. Since $A \times B$ is compactly connected, we can apply (2.2), and for each g in G ,

$$A^* \times B^* = \bigcup (A \times B)x \overline{D(g)(A_e \times B_e)}$$

is the LC-partition of $A^* \times B^*$ with respect to $(A \times B, D(g))$. Hence

$$A^*gB^* = \bigcup \xi_g((A \times B)x \overline{D(g)(A_e \times B_e)})$$

is a partition of A^*gB^* into minimal, locally compact, (A, B) -invariant subsets. This completes the proof of (a).

PROOF OF (b) AND (c). By (7) and (6), A^* and B^*/B are abelian groups. Hence $(A \times B)D(g)$ is a subgroup of $A^* \times B^*$, and the coset decomposition

$$A^* \times B^* = \bigcup x \overline{(A \times B)D(g)}$$

gives the LC-partition of $A^* \times B^*$ with respect to the pair $(A \times B, D(g))$. Hence G has an LC-partition with respect to the pair (A, B) .

Since $(\{e\} \times B)D(g)$ is a closed normal subgroup of $A^* \times B^*$, the set $(\{e\} \times B) \setminus A^* \times B^*/D(g)$ can be identified with the abelian group $L(g) = A^* \times B^*/(\{e\} \times B)D(g)$. Thus the map

$$\eta_g: A^* \times B^* \ni (\alpha, \beta) \mapsto \alpha\beta^{-1}B \in A^*gB^*/B$$

induces a homeomorphism η'_g from $L(g)$ onto A^*gB^*/B .

Let us put $(A \times B)D(g)/(\{e\} \times B)D(g) = L'(g)$. Then for each $x \in A^* \times B^*$, we put $x^* = x(\{e\} \times B)D(g) \in L(g)$, and get that

$$\eta'_g(x^*L'(g)) = \eta_g(x \overline{(A \times B)D(g)})$$

is the local closure of the orbit $A\eta_g(x)$.

If, in particular, A is a one-parameter subgroup, then $(A \times B)D(g)/(\{e\} \times B)D(g)$ is a one-parameter subgroup of $L(g)$, and is dense in $L'(g)$. Hence $L'(g)$ is either $\mathbf{R}$ or compact. This completes the proof of (b) and (c).

□

REMARK. Let L be a locally compact, σ -compact group, and G a closed

subgroup of L . Let A and B be subgroups of G . If L has an LC-family $\mathcal{F}$, then Theorem III, except for the part concerning $\mathcal{F}$ -hulls, holds for G . In fact, if P_λ is the local closure of AgB in L for $g \in G$, then $P_\lambda \cap G$ is locally compact, (A, B) -invariant, and coincides with P_λ . Hence $P_\lambda \subset G$.

3. Semi-algebraic groups. In this section we shall study subgroups of $GL(n, \mathbf{R})$. A subgroup H of $GL(n, \mathbf{R})$ is said to be *pre-algebraic* if H is an open subgroup of a suitable algebraic group. Since an algebraic group is finitely connected, so is any pre-algebraic group. Let $\mathcal{Q}$ denote the set of all pre-algebraic groups in $GL(n, \mathbf{R})$. The following theorem is known; in particular, the proof of (4) for $\mathcal{Q}$ can be found in [2].

(3.1) THEOREM. $\mathcal{Q}$ is an LC-family in $GL(n, \mathbf{R})$. Any member of $\mathcal{Q}$ is finitely connected.

For a subgroup H of $GL(n, \mathbf{R})$, let $\mathcal{Q}_e(H)$ denote the identity component of $\mathcal{Q}(H)$.

(3.2) (1) For any subgroup H of $GL(n, \mathbf{R})$, $H \cdot \mathcal{Q}_e(H) = \mathcal{Q}(H)$.

(2) If in particular H is finitely connected, then $\mathcal{Q}_e(H) = \mathcal{Q}(H_e)$.

PROOF. (1) Since $\mathcal{Q}_e(H)$ is a normal subgroup of $\mathcal{Q}(H)$, we have that $H \cdot \mathcal{Q}_e(H)$ is a subgroup of $\mathcal{Q}(H)$. Since $\mathcal{Q}_e(H)$ is contained in $H \cdot \mathcal{Q}_e(H)$, we see that $H \cdot \mathcal{Q}_e(H)$ is open in $\mathcal{Q}(H)$, and is pre-algebraic. Hence $H \cdot \mathcal{Q}_e(H) = \mathcal{Q}(H)$.

(2) It is obvious that $\overline{H}_e$ is the identity component of $\overline{H}$. Hence it reduces to (2.5). $\square$

Let H be a closed connected subgroup of $GL(n, \mathbf{R})$. Then H is normal in $\mathcal{Q}(H)$, and the factor group $\mathcal{Q}(H)/H$ is a connected abelian group by (6). Since $\mathcal{Q}(H)/H$ is a Lie group, there exist nonnegative integers k and h such that $\mathcal{Q}(H)/H = \mathbf{R}^k \times (\mathbf{R}/\mathbf{Z})^h$. Let K be a maximal compact subgroup of $\mathcal{Q}(H)$. Then KH/H is a maximal compact subgroup of $\mathcal{Q}(H)/H$; see Iwasawa [5].

In [2], the author defined H to be *semi-algebraic* if H contains all compact subgroups of $\mathcal{Q}(H)$. This is equivalent to saying that the factor group $\mathcal{Q}(H)/H$ is isomorphic with the vector group $\mathbf{R}^k$, or that $\mathcal{Q}(H)/H$ is homeomorphic with the euclidean space $\mathbf{R}^k$. Now let us extend the definition to nonconnected groups.

DEFINITION 5. A closed subgroup H of $GL(n, \mathbf{R})$ is said to be *semi-algebraic* if the factor space $\mathcal{Q}(H)/H$ is homeomorphic with a euclidean space.

REMARK AND CORRECTION. In [2], the author defined nonconnected semi-algebraic groups in a more restrictive manner. However, he later found the new definition more convenient. In the Proposition, p. 72 in [2], "and conversely." must be removed.

(3.3) Let G be a Lie group, and H a closed subgroup of G . If the factor

space G/H is connected and simply connected, then

$$H \cap G_e = H_e \quad \text{and} \quad G/G_e \cong H/H_e.$$

PROOF. Since G/H is connected, we have that $G_e H = G$ and $G/H \sim G_e/H \cap G_e$, where $\sim$ denotes the existence of a homeomorphism. Since G/H is simply connected, $H \cap G_e$ is connected and coincides with H_e . Hence $G/G_e = HG_e/G_e \cong H/H \cap G_e = H/H_e$. $\square$

(3.4) If S is semi-algebraic, then S is finitely connected and S_e is semi-algebraic, and vice versa.

PROOF. If S is semi-algebraic, then by (3.3), $S/S_e \cong \mathcal{Q}(S)/\mathcal{Q}_e(S)$ is finite and $\mathcal{Q}_e(S) \cap S = S_e$. Hence by (3.2),

$$\mathcal{Q}(S)/S = S \cdot \mathcal{Q}_e(S)/S \sim \mathcal{Q}_e(S)/\mathcal{Q}_e(S) \cap S = \mathcal{Q}(S_e)/S_e,$$

and $\mathcal{Q}(S_e)/S_e$ is homeomorphic with a euclidean space.

Conversely, suppose that S is finitely connected and S_e is semi-algebraic. Then $(\mathcal{Q}_e(S) \cap S)/S_e$ is a subgroup of $\mathcal{Q}_e(S)/S_e = \mathcal{Q}(S_e)/S_e \cong \mathbf{R}^k$. Since $(\mathcal{Q}_e(S) \cap S)/S_e$ is a finite group, it must reduce to the identity and $\mathcal{Q}_e(S) \cap S = S_e$. Then

$$\begin{aligned} \mathcal{Q}(S)/S &= S \cdot \mathcal{Q}_e(S)/S_e = S \cdot \mathcal{Q}(S_e)/S_e \sim \mathcal{Q}(S_e)/(\mathcal{Q}(S_e) \cap S) \\ &= \mathcal{Q}(S_e)/S_e \cong \mathbf{R}^k. \quad \square \end{aligned}$$

Let G be a countably connected Lie group. A subgroup H of G is said to be a *Lie subgroup* if there exists a countably connected Lie group H^* and a continuous one-one homomorphism f from H^* into G such that $f(H^*) = H$. A closed subgroup is a Lie subgroup. Let H be a Lie subgroup of G . Then the Lie group H^* is uniquely determined up to topological isomorphisms. H is called a *connected*, or a *finitely connected*, *Lie subgroup* if H^* is connected, or finitely connected, respectively. H is a connected Lie subgroup if and only if H is arcwise connected. If H_1 and H_2 are Lie subgroups and if $H_1 H_2$ is a subgroup, then $H_1 H_2$ is a Lie subgroup.

Let us denote the set of all semi-algebraic groups in $GL(n, \mathbf{R})$ by $\mathfrak{S}$.

(3.5) Let $\{S_\lambda\}$ be a subset of $\mathfrak{S}$. Then the intersection $\cap S_\lambda$ is semi-algebraic.

PROOF. Since any semi-algebraic group is finitely connected, $\mathfrak{S}$ satisfies the descending chain condition. Hence it suffices to prove that $A \cap B \in \mathfrak{S}$ for A and B in $\mathfrak{S}$. We put

$$A_1 = \mathcal{Q}(A_e), \quad B_1 = \mathcal{Q}(B_e) \quad \text{and} \quad C_1 = A_1 \cap B_1.$$

Then C_1 is pre-algebraic and is finitely connected. Therefore $A_e C_1$ is a finitely connected Lie subgroup of A_1 , and so is $A_e C_1/A_e$ in A_1/A_e . Since A_1/A_e is a vector group, so is $A_e C_1/A_e$. Thus $C_1/C_1 \cap A_e \cong A_e C_1/A_e$ is a vector group. By (3.3), $C_1 \cap A_e$ is finitely connected since C_1 is. In a similar way, $C_1/C_1 \cap$

B_e is a vector group and $C_1 \cap B_e$ is finitely connected. It, then, follows that $(C_1 \cap A_e)(C_1 \cap B_e)$ is a finitely connected Lie subgroup in C_1 , and so is $(C_1 \cap A_e)(C_1 \cap B_e)/C_1 \cap B_e$ in $C_1/C_1 \cap B_e = \mathbf{R}^k$. Hence

$$(C_1 \cap A_e)(C_1 \cap B_e)/C_1 \cap B_e \cong C_1 \cap A_e/C$$

is a vector group, where $C = A_e \cap B_e = (C_1 \cap A_e) \cap (C_1 \cap B_e)$. $C_1 \cap A_e$ being finitely connected, so is C by (3.3). Moreover, C is of finite index in $A \cap B$ since A and B are finitely connected. Therefore $A \cap B$ is finitely connected.

Next, we shall prove that $(A \cap B)_e$ is semi-algebraic. Let K be a compact connected subgroup of C_1 . Then $K \subset A_1$, and $K \subset A$. Similarly $K \subset B$. Hence $K \subset (A \cap B)_e$. That is, any maximal compact subgroup of $(C_1)_e$ is contained in $(A \cap B)_e$. It follows from $(C_1)_e$ being pre-algebraic that $(A \cap B)_e$ is semi-algebraic. $\square$

(3.6) Let A and B be in $\mathcal{S}$. Then the double coset AB is locally compact.¹

PROOF. As before, we put $\mathcal{Q}(A) = A^*$, $\mathcal{Q}(B) = B^*$ and

$$D = \{(\gamma, \gamma); \gamma \in A^* \cap B^*\} \subset A^* \times B^*.$$

Since A^*B^* is locally compact, the map

$$\xi: A^* \times B^* \ni (\alpha, \beta) \rightarrow \alpha\beta^{-1} \in A^*B^*$$

is continuous and open, and gives rise to a homeomorphism between the factor space $A^* \times B^*/D$ and A^*B^* . Hence it suffices to prove that $\xi^{-1}(AB) = (A \times B)D$ is closed in A^*B^* .

Since A , B and D are all finitely connected, $(A \times B)D$ is a finite union of the sets of the form

$$\varepsilon(A_e \times B_e)D_e\delta, \quad \varepsilon \in A \times B, \delta \in D.$$

On the other hand, $(A_e \times B_e)D_e$ is a connected Lie subgroup of $(A^* \times B^*)_e$ containing all compact subgroups, and is closed. $\square$

From (3.1), (3.4), (3.5) and (3.6), we have the following theorem:

(3.7) THEOREM. $\mathcal{S}$ is an LC-family in $\mathrm{GL}(n, \mathbf{R})$, and any member of $\mathcal{S}$ is finitely connected.

(3.8) Let φ be a rational homomorphism from an algebraic group A into $\mathrm{GL}(j, \mathbf{R})$. Let S be a semi-algebraic group in A . Then $\varphi(S)$ is semi-algebraic.

PROOF. Since S is finitely connected, so is $\varphi(S)$. Hence it suffices to prove that $\varphi(S_e)$ is semi-algebraic. Therefore, we may suppose that S is connected.

Let N be the kernel of φ restricted to $\mathcal{Q}(S)$. Then N is pre-algebraic, and is finitely connected. Hence NS/S is a finitely connected Lie subgroup of $\mathcal{Q}(S)/S$. Since $\mathcal{Q}(S)/S$ is a vector group, so is NS/S , and NS is closed and

¹(3.6) was proved in [2] under a slightly stronger condition.

connected. Hence $\varphi(S) = \varphi(NS)$ is closed in $\varphi\mathcal{Q}(S)$. Since $\varphi\mathcal{Q}(S) = \mathcal{Q}\varphi(S)$, see [3], we have

$$\mathcal{Q}\varphi(S)/\varphi(S) = \varphi\mathcal{Q}(S)/\varphi(S) \cong \mathcal{Q}(S)/NS.$$

Thus, recalling $\mathcal{Q}(S)/NS = \mathcal{Q}(S)/S/NS/S$ is a vector group, it follows that $\varphi(S)$ is semi-algebraic. $\square$

4. sa-groups in an adjoint semi-algebraic group. Let G be a connected Lie group, $\mathfrak{g}$ its Lie algebra, and let ρ denote the adjoint representation of G :

$$G \ni g \mapsto \rho(g) = \text{Ad}(g) \in \text{Ad}(\mathfrak{g}) \subset \text{GL}(\mathfrak{g}).$$

The kernel of ρ is the center Z of G . The connected Lie group G is said to be *adjoint semi-algebraic* if the adjoint group $\text{Ad}(G) = \text{Ad}(\mathfrak{g})$ is semi-algebraic. By (3.8), a connected semi-algebraic group is adjoint semi-algebraic, and the converse is given by the following:

(4.1) Let G be an adjoint semi-algebraic group. Then there exists a semi-algebraic group $G' \subset \text{GL}(n, \mathbf{R})$, for a sufficiently large n , such that G is locally isomorphic with G' .

(4.1) was proved in [2], along with (4.2).

(4.2) Let G be a connected Lie group. Then there exists an adjoint semi-algebraic group S containing G as a closed normal subgroup.

After this, we assume that G is an adjoint semi-algebraic group, $\mathfrak{g}$ its Lie algebra, ρ the adjoint representation of G and Z is the center of G .

DEFINITION 6. A subgroup H of G is said to be an *sa-group* if

(i) $\rho(H)$ is semi-algebraic, and

(ii) H is open in $\rho^{-1}\rho(H)$.

If H is an sa-group, then

(ii') H is closed and $H \supset Z_e$.

Conversely, (i) and (ii') imply (ii) obviously. Let $\mathfrak{s}$ denote the set of all sa-groups in G .

(4.3) Let S be an sa-group.

(1) If S' is an open subgroup of S , then S' is sa.

(2) If S'' contains S as a subgroup of finite index, then S'' is an sa-group.

PROOF. If S is an sa-group, then $\rho(S)$ is semi-algebraic and

$$\rho(S'') \supset \rho(S) \supset \rho(S') \supset \rho(S_e) = \rho(S'')_e.$$

Hence $\rho(S')$ is open in $\rho(S)$, and is semi-algebraic. Also $\rho(S)$ is of finite index in $\rho(S'')$, and $\rho(S'')$ is semi-algebraic.

Next, S' and S'' are closed subgroups and contain $S_e \supset Z_e$. Hence S' and S'' are sa-groups. $\square$

(4.4) If $\{S_\lambda; \lambda \in \Lambda\}$ is a nonempty subset of $\mathfrak{s}$, then $\bigcap S_\lambda \in \mathfrak{s}$.

PROOF. We put $S'_\lambda = S_\lambda Z = \rho^{-1}\rho(S_\lambda)$ for $\lambda \in \Lambda$. Then it is obvious that

$$\cap \rho(S'_\lambda) = \rho(\cap S'_\lambda) \quad \text{and} \quad \rho^{-1}\rho(\cap S'_\lambda) = \cap S'_\lambda.$$

Since $\rho(S'_\lambda) = \rho(S_\lambda)$ is semi-algebraic, so is $\cap \rho(S'_\lambda)$. Hence $\cap S'_\lambda$ is an sa-group.

Next, since S_λ is open in S'_λ , the identity components of the two groups coincide, and $(\cap S_\lambda)_e = (\cap S'_\lambda)_e$. Therefore $\cap S'_\lambda \supset \cap S_\lambda \supset (\cap S'_\lambda)_e$, and $\cap S_\lambda$ is an open subgroup of $\cap S'_\lambda$. By (4.3), $\cap S_\lambda$ is an sa-group. $\square$

By (4.4), for any subgroup H of G , there corresponds the $\mathfrak{s}$ -hull $\mathfrak{s}(H)$, the smallest sa-group containing H . Let $\mathfrak{s}_e(H)$ denote the identity component of $\mathfrak{s}(H)$.

(4.5) (1) If H is a connected subgroup of G , then

$$\mathfrak{s}(H) = (\rho^{-1}\mathfrak{S}\rho(H))_e.$$

(2) If H is a finitely connected, closed subgroup of G , then

$$\mathfrak{s}(H) = H \cdot \mathfrak{s}_e(H), \quad \mathfrak{s}_e(H) = \mathfrak{s}(H_e),$$

$\mathfrak{s}(H)$ is finitely connected, and H_e is a normal subgroup of $\mathfrak{s}(H)$.

PROOF. (1) We put $S = (\rho^{-1}\mathfrak{S}\rho(H))_e$. Then

$$\rho(S) = \mathfrak{S}\rho(H) \quad \text{and} \quad \rho^{-1}\rho(S)_e = S,$$

and S is an sa-group.

Next, suppose that T is an sa-group containing H . Then

$$\rho(T) \supset \mathfrak{S}\rho(H) \quad \text{and} \quad \rho^{-1}\rho(T) \supset \rho^{-1}\mathfrak{S}\rho(H) \supset S.$$

Since T is open in $\rho^{-1}\rho(T)$ and S is connected, we have that $T \supset S$.

(2) Let $\mathcal{K}$ be the Lie algebra of H , and let N be the normalizer of H_e in G . Then $M = \{x \in \text{GL}(\mathcal{G}); x\mathcal{K} = \mathcal{K}\}$ is an algebraic group, and $\rho(N) = \text{Ad}(\mathcal{G}) \cap M$ is a semi-algebraic group in $\text{GL}(\mathcal{G})$. Hence

$$\rho(N) \supset \mathfrak{S}\rho(H_e) \quad \text{and} \quad N = \rho^{-1}\rho(N) \supset \rho^{-1}\mathfrak{S}\rho(H_e).$$

Since $\rho(H_e)$ is a normal subgroup of $\rho(N)$, for $x \in \rho(N)$

$$\mathfrak{S}\rho(H_e) = \mathfrak{S}(x\rho(H_e)x^{-1}) = x\mathfrak{S}\rho(H_e)x^{-1},$$

and $\mathfrak{S}\rho(H_e)$ is a normal subgroup of $\rho(N)$. Hence N normalizes $\rho^{-1}\mathfrak{S}\rho(H_e)$, and its identity component $\mathfrak{s}(H_e)$. Since H is contained in N , $H \cdot \mathfrak{s}(H_e)$ is a subgroup of N . By $\mathfrak{s}(H_e) \supset H_e$, $\mathfrak{s}(H_e)$ is of finite index in $H \cdot \mathfrak{s}(H_e)$. Hence $H \cdot \mathfrak{s}(H_e)$ is an sa-group, by (4.3) (2), and $\mathfrak{s}(H) = H \cdot \mathfrak{s}(H_e)$. Thus $\mathfrak{s}(H_e)$ is a closed connected subgroup of finite index in $\mathfrak{s}(H)$, and so $\mathfrak{s}_e(H) = \mathfrak{s}(H_e)$. That $N \supset \mathfrak{s}(H)$ implies that H_e is normal in $\mathfrak{s}(H)$. $\square$

For a group L , let $[L, L]$ denote the commutator subgroup of L . If, in particular, L is a connected Lie group with Lie algebra $\mathcal{L}$, then $[L, L]$ is a connected Lie subgroup of L and the Lie algebra of $[L, L]$ is $[\mathcal{L}, \mathcal{L}]$.

(4.6) Let H be a connected Lie subgroup of G . Then $[\mathfrak{z}(H), \mathfrak{z}(H)] = [H, H]$.

PROOF. First suppose that G is a semi-algebraic group in $\mathrm{GL}(n, \mathbf{R})$. We put $S = \mathfrak{S}(H)Z$. Then $[S, S] = [\mathfrak{S}(H), \mathfrak{S}(H)] = [H, H]$. By (3.8), $\rho(S) = \rho\mathfrak{S}(H)$ is semi-algebraic. Hence $S = \rho^{-1}\rho(S)$ is an sa-group, and $S \supset \mathfrak{z}(H)$. Therefore $[S, S] \supset [\mathfrak{z}(H), \mathfrak{z}(H)] \supset [H, H]$ and $[\mathfrak{z}(H), \mathfrak{z}(H)] = [H, H]$.

Now, we shall consider the general case. By (4.1), there exists a connected semi-algebraic group G' which is locally isomorphic with G . Let us identify $\mathfrak{g}$ with the Lie algebra of G' . Let $\mathcal{H}$ be the Lie algebra of H , and let H' be the connected Lie subgroup of G' corresponding to the Lie algebra $\mathcal{H}$. Then by (4.5) (1), the Lie algebra of $\mathfrak{z}(H')$ coincides with the Lie algebra $\mathfrak{z}(\mathcal{H})$ of $\mathfrak{z}(H)$. Thus $[\mathfrak{z}(H'), \mathfrak{z}(H')] = [H', H']$ implies $[\mathfrak{z}(\mathcal{H}), \mathfrak{z}(\mathcal{H})] = [\mathcal{H}, \mathcal{H}]$, whence $[\mathfrak{z}(H), \mathfrak{z}(H)] = [H, H]$. $\square$

(4.7) If H is a finitely connected, abelian subgroup of G , then $\mathfrak{z}(H)$ is abelian.

PROOF. Because $\overline{HZ_e}$ is finitely connected and abelian, we can suppose that H is closed and $H \supset Z_e$, without loss of generality. By (4.5) (2),

$$\mathfrak{z}(H) = H \cdot \mathfrak{z}_e(H), \quad \mathfrak{z}_e(H) = \mathfrak{z}(H_e),$$

where $\mathfrak{z}_e(H)$ is abelian, by (4.6).

Let A be an abelian group in $\mathrm{GL}(n, \mathbf{R})$, and let C be the center of the centralizer of A . Then C is an abelian algebraic group and $C \supset A$. Therefore $\mathcal{Q}(A)$ is abelian, and so is $\mathfrak{S}(A)$. That is, *the $\mathfrak{S}$ -hull of any abelian group is abelian*. Since H is abelian, so is $\rho(H)$, and $\mathfrak{S}\rho(H)$ is an abelian group. Hence $[\mathfrak{z}(H), \mathfrak{z}(H)] \subset Z$.

Let $\mathcal{H}$ and $\mathfrak{z}(\mathcal{H})$ denote the Lie algebra of H and $\mathfrak{z}(H)$, respectively. $\mathfrak{z}_e(H)$ being a normal subgroup of $\mathfrak{z}(H)$, $\rho(H)$ leaves $\mathfrak{z}(\mathcal{H})$ invariant. For $h \in H$, let $\mu(h)$ denote the restriction of $\rho(h)$ to $\mathfrak{z}(\mathcal{H})$. Since $\mathfrak{z}_e(H)$ is abelian and $H_e \subset \mathfrak{z}_e(H)$, the kernel of μ contains H_e , and μ induces a representation of the finite group H/H_e . Therefore, the representation $\mu: H \rightarrow \mathrm{GL}(\mathfrak{z}(\mathcal{H}))$ is completely reducible. Owing to $\mu(H)\mathcal{H} = \mathcal{H}$, we can find a subspace (subalgebra) $\mathfrak{N}$ of $\mathfrak{z}(\mathcal{H})$ such that $\mathfrak{z}(\mathcal{H}) = \mathcal{H} \oplus \mathfrak{N}$, $\mu(H)\mathfrak{N} = \mathfrak{N}$. For any h in H , we have $(\mu(h) - 1)\mathcal{H} = 0$ because H is abelian, and $(\mu(h) - 1)\mathfrak{z}(\mathcal{H}) \subset \mathfrak{N}$.

On the other hand, for any $X \in \mathfrak{z}(\mathcal{H})$ and the real parameter t ,

$$a(t) = h(\exp tX)h^{-1}(\exp(-tX)) = \exp(t(\mu(h) - 1)X + O(t^2))$$

is a curve in Z , and belongs to Z_e . Since $Z_e \subset H$ and the tangent vector to $a(t)$ at $t = 0$ is $(\mu(h) - 1)X$, we have that $(\mu(h) - 1)X \in \mathcal{H}$, and $(\mu(h) - 1)\mathfrak{z}(\mathcal{H}) \subset \mathcal{H}$. It follows that $(\mu(h) - 1)\mathfrak{z}(\mathcal{H}) = 0$ since $\mathcal{H} \cap \mathfrak{N} = \{0\}$. Hence h commutes with every element of $\mathfrak{z}_e(H)$. Therefore $\mathfrak{z}(H) = H \cdot \mathfrak{z}_e(H)$ is abelian. $\square$

REMARK. The semi-algebraic hull of an abelian group is abelian as we saw in the proof above. But this is not true for the $\mathfrak{s}$ -hull. For example,

$$G = \left\{ g(x, y, z) = \begin{pmatrix} e^x & 0 & 0 & 0 \\ 0 & \cos x & \sin x & y \\ 0 & -\sin x & \cos x & z \\ 0 & 0 & 0 & 1 \end{pmatrix}; x, y, z \in \mathbf{R} \right\}$$

is an adjoint (semi-)algebraic group, and

$$H = \{ g(x, y, z); x \in 2\pi\mathbf{Z}, y \in \mathbf{Z}, z \in \mathbf{Z} \}$$

is an abelian subgroup of G , but $\mathfrak{s}(H) = G$ is not abelian.

Thus we have

(4.8) THEOREM. *Let G be an adjoint semi-algebraic group, and let $\mathfrak{s}$ be the totality of sa-groups in G . Then $\mathfrak{s}$ is an LC-family.*

ADDED IN PROOF. The author learned from Philip Green that he had proved in his unpublished paper that any connected Lie group can be embedded as a closed normal subgroup in a suitable Lie group whose adjoint group is pre-algebraic. By his theorem, the main results of this paper can be established without semi-algebraic groups. But the author thinks the theory of semi-algebraic groups itself has some significance, so he leaves the paper in the original form.

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